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A FRAMEWORK FOR NONLINEAR SHELLS BASED ON GENERALIZED STRESS AND STRAIN MEASURES

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Abstract—A very general class of shell theories is established, based on a unified variational principle and on expansions with respect to biorthogonal function systems. The shell models derived are able to accommodate the flow theory of large strain thermoelastoplasticity including the effects due to unloading. The specific case of a compressible neo-Hookean material with a von Mises yield condition is discussed in detail. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The development of shell theories has been marked by two competing approaches. There are, on the one hand, the "degenerated solid formulations", see, e.g., the treatments in the books by Crisfield (1991), or by Zienkiewicz and Taylor (1991), which offer the advantages of great flexibility and easy numerical implementation. On the other hand we have the "stress resultant based formulations" which comprise all classical shell and plate theories and have been revived by Simo and his coworkers in a series of papers; see, e.g., Simo and Fox (1989), Simo and Kennedy (1992), compare also the account in the book by Antman (1995). They lead to analytical models which inherit the structure of the underlying three-dimensional theories and have the benefit of being accessible to methods of mathematical analysis yielding existence and uniqueness or bifurcation results. Also, the quantities involved can be better interpreted from a mechanical point of view.

Considering elastoplastic shells now, we have to face another difficulty. Because of history-dependent inelastic deformations, the stresses become history-dependent themselves and show complicated behavior across the shell thickness, e.g., during buckling we typically have plastic loading and elastic unloading in a single (!) cross section. This seems to leave us with the "degenerated solid approach" as the only way to go. The features mentioned above, however, make it still very desirable to have "stress resultant based models". Simo and Kennedy (1992), and Gilbert and Hackl (1995), have made efforts in this direction. Because of restrictions of the constitutive models used in those papers, however, the theories developed are not able to capture effects involving simultaneous loading and unloading.

Similar problems arise to a smaller extent when dealing with nonlinearly elastic materials.

The purpose of this paper is to present a class of shell theories which offers the following benefits:

(i) It contains practically all existing "degenerated solid" as well as "stress resultant based" theories as special cases, thus combining the advantages of those.
(ii) It allows us to treat simultaneous loading and unloading in a single cross section.
(iii) It can be adapted in a flexible and economical way to essentially any constitutive model of nonlinear elasticity or elastoplasticity.
(iv) It reflects the structure of the underlying three-dimensional theory.

The theory proposed relies on two fundamental ideas: first of all, the power series expansions across the shell thickness of the quantities involved developed in the works by Naghdi (1972), and Dikmen (1982), are generalized to expansions with respect to arbitrary biorthogonal function systems. Only in this way can we obtain enough flexibility to treat elastoplastic problems with their unilateral constraints.

Secondly we substitute those expansions into a novel variational principle developed by Hackl (1995), which constitutes an extension of the well known Hu-Washizu-principle to elastoplastic materials. This allows, on the one hand, a rational derivation of the two-dimensional theory from the three-dimensional one. On the other hand different expansions can be introduced for the kinematical and constitutive quantities, thus allowing very economical modelling. A Hu-Washizu-principle in connection with inelastic shells has already been used by Mukherjee and Kollman (1985).

An important point is that the paper does not aim to give an asymptotic analysis in the spirit of Ciarlet (1980), i.e., to investigate the limit behavior as the shell thickness tends to zero. The reason for this is that, for nonlinear materials, only the tangent modulus at zero strain is retained by the asymptotics, as shown by Davet (1986), which leads, especially in the elastoplastic case, to mechanically unrealistic models. For such materials there is typically a coupling of effects which are scaled with different powers of the shell thickness. This means we have to work with (generally) thin but not infinitely thin bodies. Attempts for asymptotic theories for elastoplastic shells have been made by Destuynder and Nguyen (1985), and Gilbert and Hackl (1995).

The course of the paper will be as follows: it starts with a brief exposition of shell kinematics in Section 2. In Section 3 we introduce the constitutive theory of thermo-elastoplasticity and the related unified variational principle. In Section 4 we discuss three fundamental assumptions distinguishing shells from general three-dimensional bodies. In Section 5 we introduce biorthogonal function systems. In Section 6 we perform the reduction to a two-dimensional theory. Section 7 deals with the unilateral constraints imposed by elastoplasticity. In Section 8 we derive the expression for the elastoplastic tangent operator. Finally, in order to demonstrate the capabilities of the theoretical framework developed, we specify it to a model of large strain elastoplasticity in Section 9. Conclusions are drawn in Section 10.

2. FUNDAMENTAL KINEMATICS

Notation: the dot product of a covariant vector (linear form) $\boldsymbol{\alpha} = (\alpha_i)$ and a contravariant vector $\mathbf{v} = (v^i)$ is defined by $\boldsymbol{\alpha} \cdot \mathbf{v} := \boldsymbol{\alpha}(\mathbf{v}) = \alpha_i v^i$, where we have adopted the usual summation convention. In a similar manner we define the dot products of a tensor and a vector, $\mathbf{T} \cdot \mathbf{v} := (T^i_j v^j)$, and of two tensors, $(\mathbf{A} \cdot \mathbf{B})^i_j := A^i_k B^k_j$. The inner product of two tensors is given by $\mathbf{S} : \mathbf{T} := S^i_j T^j_i = \mathrm{tr}(\mathbf{S}^{\mathrm{T}} \cdot \mathbf{T})$, where the trace $\mathrm{tr}(\mathbf{T})$ of a mixed variant tensor $\mathbf{T} = (T^i_j) : \mathscr{V} \to \mathscr{V}$ is the sum of its eigenvalues; in components $\mathrm{tr}(\mathbf{T}) = T^i_i$.

For a detailed discussion, especially of the differential geometric meaning of the various operations explained above, see the exposition in Hackl (1995).

Let $\omega \subset \mathbb{R}^2$ be an open set with boundary γ. Then a shell is represented by its reference configuration $\Omega := \omega \times (-\varepsilon, +\varepsilon)$, $\varepsilon > 0$, equipped with coordinates $\{\zeta^i\}_{i=1,2,3}$. We call ω the reference surface of the shell, $\Gamma_o := \gamma \times (-\varepsilon, +\varepsilon)$ the edge of the shell and $\Gamma_\pm := \omega \times \{\pm\varepsilon\}$ the faces of the shell.

Further, let $\mathscr{S} \subset \mathbb{R}^3$ be an open set with coordinate system $\{x^i\}_{i=1,2,3}$ and metric tensor $\boldsymbol{g} = (g_{ij})$, which throughout this paper we will assume constant (independent of x^i and time t).

An actual configuration $\hat{\phi}(\Omega) \in \mathscr{S}$ of the shell is given by an orientation preserving diffeomorphism $\hat{\phi} : \Omega \to \mathscr{S}$.† $\hat{\phi}$ can be uniquely represented in the form

$$\hat{\phi}(\boldsymbol{\zeta}, \xi) = \mathbf{r}(\boldsymbol{\zeta}) + \mathbf{d}(\boldsymbol{\zeta}, \xi) \quad (1)$$

† Of course, from a differential geometric point of view, $\hat{\phi}$ is nothing else than a map for a three dimensional manifold. A shell can be any three dimensional manifold which can be embedded into $\mathbb{R}^3$, also a multiply connected one. So the description given above has to be understood in a local sense, i.e., you might need more than one map $\hat{\phi}$ in order to represent a shell.

with

$$\int_{-\varepsilon}^{+\varepsilon} \mathbf{d}(\boldsymbol{\zeta}, \xi)\, \mathrm{d}\xi = 0, \tag{2}$$

where $\boldsymbol{\zeta} = (\zeta^\alpha)_{\alpha=1,2}$ and $\xi = \zeta^3$. Note that, in general, $\mathbf{d}(\boldsymbol{\zeta}, 0) \neq 0$.

$\mathbf{r} = (r^i)$ denotes the position vector of the reference surface of the shell, $\mathbf{d} = (d^i)$, the shell director.

Let $\mathscr{B} \subset \mathbb{R}^3$ be another open set with coordinate system $\{X^I\}_{I=1,2,3}$ and metric tensor $\mathbf{G} = (G_{IJ})$. The initial configuration ${}^0\phi(\Omega)$ of the shell is given by an orientation preserving diffeomorphism ${}^0\phi : \Omega \to \mathscr{B}$ of the form

$${}^0\phi(\boldsymbol{\zeta}, \xi) = {}^0\mathbf{r}(\boldsymbol{\zeta}) + {}^0\mathbf{d}(\boldsymbol{\zeta}, \xi). \tag{3}$$

In contrast to $\hat{\phi}$ we will consider ${}^0\phi$ to be constant with respect to time t.† Without restriction the initial director ${}^0\mathbf{d}$ can be assumed to be perpendicular to the initial reference surface ${}^0\phi(\omega)$ and to depend linearly on ξ, i.e., let

$${}^0\mathbf{a}_\alpha := {}^0\mathbf{r}_{,\alpha}\text{‡} \tag{4}$$

be the natural basis vectors of the tangent space $T\mathscr{B}$ along the coordinate lines ζ^α, then (identifying $\mathscr{B}$ and $T\mathscr{B}$) ${}^0\mathbf{d}$ can be defined by

$${}^0\mathbf{d}(\boldsymbol{\zeta}, \xi) := \frac{h(\boldsymbol{\zeta})}{2\varepsilon}\, \xi \frac{{}^0\mathbf{a}_1(\boldsymbol{\zeta}) \wedge {}^0\mathbf{a}_2(\boldsymbol{\zeta})}{\|{}^0\mathbf{a}_1(\boldsymbol{\zeta}) \wedge {}^0\mathbf{a}_2(\boldsymbol{\zeta})\|}, \tag{5}$$

where $h(\boldsymbol{\zeta})$ denotes the thickness of the shell. The vector product in $T\mathscr{B}$ is given by $(\mathbf{v} \wedge \mathbf{w})^I := (G^{-1})^{IJ} \varepsilon_{JKL} v^K w^L$, where ε_{JKL} denotes the totally antisymmetric tensor of order three. ε_{JKL} is uniquely determined by the property $(G^{-1})^{IJ} \varepsilon_{IKL} \varepsilon_{JMN} = G_{KM} G_{LN} - G_{KN} G_{LM}$. The norm of a vector in $T\mathscr{B}$ is defined by $\|\mathbf{v}\| := \sqrt{\mathbf{v} \cdot \mathbf{G} \cdot \mathbf{v}}$.

A deformation of the shell is then given by a mapping

$$\boldsymbol{\phi} := \hat{\boldsymbol{\phi}} \circ {}^0\boldsymbol{\phi}^{-1} : \mathscr{B} \to \mathscr{S}. \tag{6}$$

The deformation gradient§ $\mathbf{F} : T\mathscr{B} \to T\mathscr{S}$ is the tangent map of $\boldsymbol{\phi}$, given by

$$\mathbf{F} = (F^i_J) := \nabla \boldsymbol{\phi} = (\phi^i_{,J}). \tag{7}$$

It holds that

$$\mathbf{F} = \hat{\mathbf{F}} \cdot {}^0\mathbf{F}^{-1}, \tag{8}$$

where $\hat{\mathbf{F}} : T\Omega \to T\mathscr{S}$ and ${}^0\mathbf{F} : T\Omega \to T\mathscr{B}$, respectively, are the tangent maps of $\hat{\boldsymbol{\phi}}$, respectively, ${}^0\boldsymbol{\phi}$. We have

$$\hat{F}^i_\alpha = r^i_{,\alpha} + d^i_{,\alpha} \tag{9}$$

† Here, and further on, we will suppress any explicit time dependency of the quantities present.

‡ As usual, here and subsequently, Greek indices range over the set $\{1, 2\}$ while Latin indices range over the set $\{1, 2, 3\}$. A comma in front of an index means partial differentiation with respect to the corresponding variable.

§ Note that $\mathbf{F}$ is not a gradient in the sense of vector analysis because ϕ connects different manifolds rather than being a scalar field on a single manifold.

and

$$\hat{F}^i_3 = d^i_{,3}. \tag{10}$$

A central feature of finite strain elastoplasticity is the multiplicative split of the deformation gradient into a plastic and an elastic part, first introduced by Lee (1969). We will encounter this formulation again in Section 9.2. Let us therefore introduce an open set $\bar{\mathscr{B}} \subset \mathbb{R}^3$ with coordinate system $\{\bar{X}^{\bar{I}}\}_{I=1,2,3}$ and metric tensor $\bar{\mathbf{G}} = (\bar{G}_{I\bar{J}})$. We call $\bar{\mathscr{B}}$ the intermediate configuration space. We assume the following decomposition:

$$\mathbf{F} = {}^e\mathbf{F} \cdot {}^p\mathbf{F} = ({}^eF^i_{\bar{K}}\,{}^pF^{\bar{K}}{}_J), \tag{11}$$

with ${}^p\mathbf{F} : T\mathscr{B} \to T\bar{\mathscr{B}}$ and ${}^e\mathbf{F} : T\bar{\mathscr{B}} \to T\mathscr{S}$. From (8) and (11) it follows that

$$\hat{\mathbf{F}} = {}^e\mathbf{F} \cdot {}^p\hat{\mathbf{F}}, \tag{12}$$

where ${}^p\hat{\mathbf{F}}$ is given by

$${}^p\hat{\mathbf{F}} := {}^p\hat{\mathbf{F}} \cdot {}^0\mathbf{F}. \tag{13}$$

Remark: note that ${}^e\mathbf{F}$ and ${}^p\mathbf{F}$ are, in general, incompatible, i.e., they are not tangent maps of any deformation fields. In the same way, ${}^p\hat{\mathbf{F}}$ is incompatible, i.e., it is generally not possible to compensate a plastic deformation by choosing a new initial configuration. This corresponds, of course, to the fact that residual stresses are built up in a body by plastic loading and subsequent elastic unloading.

3. GENERAL THERMOELASTOPLASTIC THEORY

3.1. *Unified thermoelastoplastic functional*

In Hackl (1995), it is shown that all state equations of thermoelastoplasticity can be obtained by variation of a unified thermoelastoplastic functional which, in our case, assumes the following form:

$$\begin{aligned}
\dot{\Pi}_{UTEP} = \int_{{}^0\boldsymbol{\phi}(\Omega)} \Bigg[& \frac{\partial W}{\partial \mathbf{E}} : \dot{\mathbf{E}} + \frac{\partial W}{\partial \eta}\dot{\eta} + \frac{\partial W}{\partial \mathbf{P}} : \dot{\mathbf{P}} + \frac{\partial W}{\partial {}^p\eta}{}^p\dot{\eta} \\
& - \dot{\mathbf{S}} : (\mathbf{E} - \tfrac{1}{2}(\nabla\boldsymbol{\phi}^{\mathrm{T}} \cdot \mathbf{g} \cdot \nabla\boldsymbol{\phi} - \mathbf{G})) - \mathbf{S} : (\dot{\mathbf{E}} - \mathrm{sym}(\nabla\dot{\boldsymbol{\phi}}^{\mathrm{T}} \cdot \mathbf{g} \cdot \nabla\boldsymbol{\phi})) \\
& - \theta\dot{\eta} - \dot{\theta}\eta + \mathbf{Q} : \dot{\mathbf{P}} + {}^p\theta\,{}^p\dot{\eta} - \dot{\lambda}\Phi - \dot{\mathbf{f}} \cdot \mathbf{g} \cdot \phi - \mathbf{f} \cdot \mathbf{g} \cdot \dot{\boldsymbol{\phi}} \Bigg] \mathrm{d}V \\
& - \int_{{}^0\boldsymbol{\phi}(\Gamma_{\cdot} \cup \Gamma_{\cdot})} [\dot{\mathbf{t}} \cdot \mathbf{g} \cdot \boldsymbol{\phi} + \mathbf{t} \cdot \mathbf{g} \cdot \dot{\boldsymbol{\phi}}] \,\mathrm{d}S \\
& - \int_{{}^0\boldsymbol{\phi}(\Gamma_u)} [\dot{\mathbf{h}} \cdot \mathbf{g} \cdot (\boldsymbol{\phi} - \bar{\boldsymbol{\phi}}) + \mathbf{h} \cdot \mathbf{g} \cdot (\dot{\boldsymbol{\phi}} - \dot{\bar{\boldsymbol{\phi}}})] \,\mathrm{d}S,
\end{aligned} \tag{14}$$

where the symmetric part of a tensor $\mathbf{T} = (T_{ij})$ is given by $\mathrm{sym}(\mathbf{T}) := \frac{1}{2}(\mathbf{T} + \mathbf{T}^{\mathrm{T}}) = (\frac{1}{2}(T_{ij} + T_{ji}))$.

$\dot{\Pi}_{UTEP}$ is a functional with variables $\{\mathbf{E}, \eta, \mathbf{S}, \boldsymbol{\phi}, \mathbf{h}, \mathbf{Q}, {}^p\theta, \dot{\mathbf{E}}, \dot{\eta}, \dot{\mathbf{S}}, \dot{\boldsymbol{\phi}}, \dot{\mathbf{h}}, \dot{\mathbf{P}}, {}^p\dot{\eta}\}$ and constant parameters $\{\mathbf{P}, {}^p\eta, \mathbf{f}, \mathbf{t}, \bar{\boldsymbol{\phi}}, \theta, \dot{\mathbf{f}}, \dot{\mathbf{t}}, \dot{\bar{\boldsymbol{\phi}}}, \dot{\lambda}, \dot{\theta}\}$. Let us explain the various quantities involved in (14) by deriving its state equations:

$W = \check{W}(\mathbf{E}, \eta, \mathbf{P}, {}^p\eta)$ is the internal energy, where $\mathbf{E} = (E_{IJ})$ is the Green-Lagrange strain tensor, η denotes entropy per unit reference volume, $\mathbf{P}$ is an array of internal variables comprising especially the plastic strain ${}^p\mathbf{F}$ but also, e.g., hardening parameters, and ${}^p\eta$ is an additional internal variable which describes a plastic entropy measuring configurational change within the body (see Simo and Miehe, 1992, cp. also Hackl, 1995). Variation with respect to $\dot{\mathbf{E}}$ and $\dot{\mathbf{P}}$, respectively, gives the constitutive laws

$$\mathbf{S} := \frac{\partial W}{\partial \mathbf{E}}, \quad \mathbf{Q} := -\frac{\partial W}{\partial \mathbf{P}}, \tag{15}$$

identifying $\mathbf{S} = (S^{IJ})$ as the second Piola-Kirchhoff stress tensor and $\mathbf{Q}$ as thermodynamically associated forces to $\mathbf{P}$. Variation with respect to $\dot{\eta}$ and ${}^p\dot{\eta}$, respectively, yields temperature θ and a "plastic temperature" ${}^p\theta$:

$$\theta = \frac{\partial W}{\partial \eta}, \quad {}^p\theta = -\frac{\partial W}{\partial\, {}^p\eta}. \tag{16}$$

Note that if W depends on η and ${}^p\eta$ via an elastic entropy ${}^e\eta = \eta - {}^p\eta$, as assumed in Simo and Miehe (1992), then ${}^p\theta = \theta$.

Variation with respect to $\mathbf{S}$ gives the expression for the strain

$$\mathbf{E} = \tfrac{1}{2}(\nabla\boldsymbol{\phi}^{\mathrm{T}} \cdot \mathbf{g} \cdot \nabla\boldsymbol{\phi} - \mathbf{G}). \tag{17}$$

Variation with respect to $\dot{\boldsymbol{\phi}}$ and carrying out the usual partial integrations gives the equilibrium conditions

$$\nabla \cdot (\nabla\boldsymbol{\phi} \cdot \mathbf{S}) + \mathbf{f} = 0, \tag{18}$$

where the divergence $\boldsymbol{\alpha} = \nabla \cdot \mathbf{T}$ of a tensor $\mathbf{T} = T_i^J$ is given by $\alpha_i = T_{i,J}^J$ and $\mathbf{f} = (f^i)$ denotes a body force, and the boundary conditions

$$\mathbf{t} = \nabla\boldsymbol{\phi} \cdot \mathbf{S} \cdot \mathbf{n} \quad \text{on} \quad {}^0\phi(\Gamma_+ \cup \Gamma_-), \tag{19}$$

$$\mathbf{h} = \nabla\boldsymbol{\phi} \cdot \mathbf{S} \cdot \mathbf{n} \quad \text{on} \quad {}^0\phi(\Gamma_u), \tag{20}$$

where $\mathbf{n} = (n_I)$ denotes the outer normal on ${}^0\boldsymbol{\phi}(\Gamma_+ \cup \Gamma_-)$. (19) and (20) identify $\mathbf{t} = (t^i)$ and $\mathbf{h} = (h^i)$ as surface tractions.

Variation with respect to $\mathbf{h}$ yields the boundary conditions

$$\boldsymbol{\phi} = \bar{\boldsymbol{\phi}} \quad \text{on} \quad {}^0\boldsymbol{\phi}(\Gamma_u). \tag{21}$$

Finally, variation with respect to $\mathbf{Q}$ and ${}^p\theta$, respectively, gives the flow rules (evolution laws).

$$\dot{\mathbf{P}} = \dot{\lambda}\frac{\partial \Phi}{\partial \mathbf{Q}}, \quad {}^p\dot{\eta} = \dot{\lambda}\frac{\partial \Phi}{\partial\, {}^p\theta}, \tag{22}$$

where $\Phi = \bar{\Phi}(\mathbf{Q}, {}^p\theta, \mathbf{P}, {}^p\eta)$ is the yield function.

This defines the material under consideration as a so called generalized standard medium, see Hackl (1995). Eqn (22) is also referred to as the generalized principle of maximum plastic dissipation. $\dot{\lambda}$ is called the plastic consistency parameter. The variations are subject to the subsidiary conditions

$$\Phi \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda}\Phi = 0. \tag{23}$$

Variation with respect to $\{\mathbf{E}, \eta, \mathbf{S}, \boldsymbol{\phi}, \mathbf{h}\}$ now yields nothing but the time derivatives of (15)–(22).

3.2. *Pull back to the reference configuration*

In order to be able to perform a reduction to a two dimensional theory we have to pull back the variational principle described above from ${}^0\boldsymbol{\phi}(\Omega)$ to Ω. For this purpose we need the volume element in ${}^0\boldsymbol{\phi}(\Omega)$

$$\mathrm{d}V = {}^0j\,\mathrm{d}\xi\,\mathrm{d}\zeta^1\,\mathrm{d}\zeta^2 =: {}^0j\,\mathrm{d}\xi\,\mathrm{d}A, \tag{24}$$

with the Jacobian

$${}^0j = \sqrt{\det({}^0\mathbf{F}^{\mathrm{T}}\cdot\mathbf{G}\cdot{}^0\mathbf{F})}. \tag{25}$$

The area element along a coordinate surface ${}^0\boldsymbol{\phi}(\zeta, \xi = \text{const.})$ is given by

$$\mathrm{d}S = {}^0\bar{j}\,\mathrm{d}A, \tag{26}$$

with

$${}^0\bar{j} = \sqrt{\det({}^0\boldsymbol{\phi}_{,\alpha}\cdot\mathbf{G}\cdot{}^0\boldsymbol{\phi}_{,\beta})} = \|{}^0\boldsymbol{\phi}_{,1}\wedge{}^0\boldsymbol{\phi}_{,2}\|. \tag{27}$$

Finally, the area element along ${}^0\boldsymbol{\phi}(\Gamma_\sigma)$ is given by

$$\mathrm{d}S = {}^0\tilde{j}\,\mathrm{d}\xi\,\mathrm{d}s, \tag{28}$$

$\mathrm{d}s$ being the infinitesimal arc length along γ and ${}^0\tilde{j}$ given by

$${}^0\tilde{j} = \|{}^0\mathbf{d}_{,3}\wedge{}^0\boldsymbol{\phi}_{,s}\|, \tag{29}$$

where ${}^0\boldsymbol{\phi}_{,s}$ denotes the derivative of ${}^0\boldsymbol{\phi}$ in the direction tangent to γ.

Remark: the determinants occurring in formulae (25) and (27) are those of matrices of rank three and two defined by $\det(\mathbf{A}) := \frac{1}{6}\varepsilon_{ijk}\varepsilon_{lmn}A_{il}A_{jm}A_{kn}$ and $\det(\mathbf{A}) := \frac{1}{2}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}A_{\alpha\gamma}A_{\beta\delta}$, respectively. Here, ε_{ijk} and $\varepsilon_{\alpha\beta}$ are the total antisymmetric tensors of order three and two, respectively, in the reference configuration.

Note that those definitions rely heavily on the Cartesian structure of the reference configuration which it inherits as a subset of $\mathbb{R}^3$. The obvious sign of this fact is the summation being performed over lower indices only. A covariant definition would describe the determinant as the product of the eigenvalues of a mixed variant tensor, cf. the discussion in Hackl (1995).

It has been stated in many locations, however, that rational mechanics should refrain from using Cartesian structures because of the danger of confusing different types of quantities, see, e.g., Marsden and Hughes (1983) or Truesdell and Noll (1965). Considering shell theories the situation is different. We have to find a mean to decide what should be the "middle surface" of the shell and what should be the "fibers". This is done by using a differential geometric "map" with its Cartesian coordinate structure.

Now we are able to introduce new quantities defined on Ω by pulling back (cp. Marsden and Hughes, 1983) the original ones:

$$\hat{W} = {}^0jW,\quad \hat{\eta} = {}^0j\eta,\quad {}^p\hat{\eta} = {}^0j\,{}^p\eta,\quad \hat{\dot{\lambda}} = {}^0j\dot{\lambda},\quad \hat{\mathbf{f}} = {}^0j\mathbf{f}, \tag{30}$$

$$\hat{\mathbf{E}} = \boldsymbol{\phi}^*\mathbf{E} = {}^0\mathbf{F}^T\cdot\mathbf{E}\cdot{}^0\mathbf{F},\quad \hat{\mathbf{P}} = {}^0\boldsymbol{\phi}^*\mathbf{P},\dagger \tag{31}$$

† Especially ${}^p\hat{\mathbf{F}} = {}^0\phi^{*p}\mathbf{F} = {}^p\mathbf{F}\cdot{}^0\mathbf{F}$.

$$\hat{\mathbf{S}} = {}^0j^0\boldsymbol{\phi}^*\mathbf{S} = {}^0j^0\mathbf{F}^{T-1}\cdot\mathbf{S}\cdot{}^0\mathbf{F}^{-1}, \quad \hat{\mathbf{Q}} = {}^0j^0\boldsymbol{\phi}^*\mathbf{Q}, \tag{32}$$

$$\hat{\mathbf{t}}_\pm = {}^0\bar{j}(\zeta, \pm\varepsilon)\mathbf{t}(\zeta, \pm\varepsilon), \quad \hat{\mathbf{h}} = {}^0\bar{j}\mathbf{h}. \tag{33}$$

Substituting (30)–(33) into (14), the unified thermoelastoplastic functional becomes

$$\begin{aligned}\dot{\Pi}_{UTEP} = \int_\Omega \Bigg[& \frac{\partial\hat{\mathbf{W}}}{\partial\hat{\mathbf{E}}} : \dot{\hat{\mathbf{E}}} + \frac{\partial\hat{W}}{\partial\hat{\eta}}\dot{\hat{\eta}} + \frac{\partial\hat{W}}{\partial\hat{\mathbf{P}}} : \dot{\hat{\mathbf{P}}} + \frac{\partial\hat{W}}{\partial^p\eta}{}^p\dot{\hat{\eta}} \\ & -\dot{\hat{\mathbf{S}}} : (\hat{\mathbf{E}} - \tfrac{1}{2}(\nabla\hat{\boldsymbol{\phi}}^T\cdot\mathbf{g}\cdot\nabla\hat{\boldsymbol{\phi}} - \nabla^0\boldsymbol{\phi}^T\cdot\mathbf{G}\cdot\nabla^0\boldsymbol{\phi})) \\ & -\hat{\mathbf{S}} : (\dot{\hat{\mathbf{E}}} - \mathrm{sym}(\nabla\dot{\hat{\boldsymbol{\phi}}}^T\cdot\mathbf{g}\cdot\nabla\hat{\boldsymbol{\phi}})) \\ & -\theta\dot{\hat{\eta}} - \dot{\theta}\hat{\eta} + \hat{\mathbf{Q}} : \dot{\hat{\mathbf{P}}} + {}^p\theta\,{}^p\dot{\hat{\eta}} - \dot{\hat{\lambda}}\Phi - \dot{\hat{\mathbf{f}}}\cdot\mathbf{g}\cdot\hat{\boldsymbol{\phi}} - \hat{\mathbf{f}}\cdot\mathbf{g}\cdot\dot{\hat{\boldsymbol{\phi}}}\Bigg] \mathrm{d}\xi\,\mathrm{d}A \\ & -\int_{\Gamma_+} [\dot{\hat{\mathbf{t}}}_+\cdot\mathbf{g}\cdot\hat{\boldsymbol{\phi}} + \hat{\mathbf{t}}_+\cdot\mathbf{g}\cdot\dot{\hat{\boldsymbol{\phi}}}]\,\mathrm{d}A - \int_{\Gamma_-} [\dot{\hat{\mathbf{t}}}_-\cdot\mathbf{g}\cdot\hat{\boldsymbol{\phi}} + \hat{\mathbf{t}}_-\cdot\mathbf{g}\cdot\hat{\boldsymbol{\phi}}]\,\mathrm{d}A \\ & -\int_{\Gamma_\sigma} [\dot{\hat{\mathbf{h}}}\cdot\mathbf{g}\cdot(\hat{\boldsymbol{\phi}} - \bar{\hat{\boldsymbol{\phi}}}) + \hat{\mathbf{h}}\cdot\mathbf{g}\cdot(\dot{\hat{\boldsymbol{\phi}}} - \dot{\bar{\hat{\boldsymbol{\phi}}}})]\,\mathrm{d}\xi\,\mathrm{d}s. \end{aligned} \tag{34}$$

$\dot{\Pi}_{UTEP}$ is now equipped with variables $\{\hat{\mathbf{E}}, \hat{\eta}, \hat{\mathbf{S}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{h}}, \hat{\mathbf{Q}}, {}^p\theta, \dot{\hat{\mathbf{E}}}, \dot{\hat{\eta}}, \dot{\hat{\mathbf{S}}}, \dot{\hat{\boldsymbol{\phi}}}, \dot{\hat{\mathbf{h}}}, \dot{\hat{\mathbf{P}}}, {}^p\dot{\hat{\eta}}\}$ and constant parameters $\{\hat{\mathbf{P}}, {}^p\hat{\eta}, \hat{\mathbf{f}}, \mathbf{t}_\pm, \bar{\hat{\boldsymbol{\phi}}}, \theta, \dot{\hat{\mathbf{f}}}, \dot{\hat{\mathbf{t}}}_\pm, \dot{\bar{\hat{\boldsymbol{\phi}}}}, \dot{\hat{\boldsymbol{\lambda}}}, \dot{\theta}\}$. Of course, the newly defined quantities will satisfy the same state equations as the old ones.

4. THREE FUNDAMENTAL ASSUMPTIONS FOR SHELL THEORIES

We will introduce now three *a priori* assumptions. None of those is actually a mathematical necessity but they are meaningful from a mechanical point of view and will allow us to significantly simplify the formulation of shell theories.

First fundamental assumption for shell theories

We assume that (approximately) the external forces $\hat{\mathbf{f}}$ and $\hat{\mathbf{t}}_\pm$ perform work only via the deflection of the reference surface $\mathbf{r}$, i.e., it holds:

$$\text{(shell 1)} \quad \int_\Omega \hat{\mathbf{f}}\cdot\mathbf{d}\,\mathrm{d}\xi\,\mathrm{d}A = \text{const.}, \quad \int_{\Gamma_\pm} \hat{\mathbf{t}}_\pm\cdot\mathbf{d}\,\mathrm{d}A = \text{const.}$$

From a mechanical point of view this means that we will regard any substantial external work related to the director $\mathbf{d}$, i.e., to an intrinsic deformation of the shell, as a fundamentally three dimensional effect which cannot be addressed by a two dimensional shell theory.

The second assumption is the analog to the first one for the internal forces.

Second fundamental assumption for shell theories

We assume that (approximately) the stresses (internal forces) $\hat{\mathbf{S}}$ perform work only via the deflection of the reference surface $\mathbf{r}$, i.e., it holds:

$$\text{(shell 2)} \quad \int_\Omega \hat{\mathbf{S}} : (\nabla \mathbf{d}^{\mathrm{T}} \cdot \mathbf{g} \cdot \nabla \mathbf{d}) \, \mathrm{d}\xi \, \mathrm{d}A = \text{const.} = \int_\Omega \hat{\mathbf{S}} : (\nabla^0 \mathbf{d}^{\mathrm{T}} \cdot \mathbf{G} \cdot \nabla^0 \mathbf{d}) \, \mathrm{d}\xi \, \mathrm{d}A.$$

At this point, substitution of (shell 2) into (34) would yield $\hat{E}_{33} = 0$. Since one of our goals is to include large strains and therefore the possibility of significant changes of the shell thickness into our considerations, this is not acceptable. In order to avoid this problem we still have to introduce a third assumption.

Third fundamental assumption for shell theories

We assume that (approximately) the stresses (internal forces) $\hat{\mathbf{S}}$ perform no work via change of the shell thickness, i.e., it holds:

$$\text{(shell 3)} \quad \int_\Omega \hat{S}^{33} \hat{E}_{33} \, \mathrm{d}\xi \, \mathrm{d}A = \text{const.}$$

Before going on we have to introduce some additional notation employing the Cartesian structure of the reference configuration.

For contra-respectively covariant symmetric tensors (matrices) of rank three, $\hat{\mathbf{E}} = (\hat{E}_{ij})$ and $\hat{\mathbf{S}} = (\hat{S}^{ij})$ on $T\Omega$, we define matrices and vectors of rank two by

$$\check{\mathbf{E}} = (\check{E}_{\alpha\beta}) := (\hat{E}_{\alpha\beta}), \quad \check{\mathbf{S}} = (\check{S}^{\alpha\beta}) := (\hat{S}^{\alpha\beta}), \tag{35}$$

$$\tilde{\mathbf{E}} = (\tilde{E}_\alpha) := (\hat{E}_{\alpha 3}) = (\hat{E}_{3\alpha}), \quad \tilde{\mathbf{S}} = (\tilde{S}^\alpha) := (\hat{S}^{\alpha 3}) = (\hat{S}^{3\alpha}). \tag{36}$$

In particular, it holds that

$$\hat{\mathbf{S}} : \hat{\mathbf{E}} = \check{\mathbf{S}} : \check{\mathbf{E}} + 2\tilde{\mathbf{S}} \cdot \tilde{\mathbf{E}} + \hat{S}^{33} \hat{E}_{33}. \tag{37}$$

From (shell 3) we immediately obtain

$$\frac{\partial \hat{W}}{\partial \hat{E}_{33}} = 0. \tag{38}$$

Substitution of (1), (shell 1), (shell 2) and their time derivatives into (34) and elimination of $\hat{E}_{33}$ and $\dot{\hat{E}}_{33}$ via (38) and its time derivative gives

$$\begin{aligned}
\dot{\Pi}_{UTEP} = \int_\Omega \Bigg[& \frac{\partial \hat{W}}{\partial \check{\mathbf{E}}} : \dot{\check{\mathbf{E}}} + \frac{\partial \hat{W}}{\partial \tilde{\mathbf{E}}} \cdot \dot{\tilde{\mathbf{E}}} + \frac{\partial \hat{W}}{\partial \hat{\eta}} \dot{\hat{\eta}} + \frac{\partial \hat{W}}{\partial \hat{\mathbf{P}}} : \dot{\hat{\mathbf{P}}} + \frac{\partial \hat{W}}{\partial^p \eta} {}^p\dot{\hat{\eta}} \\
& - \dot{\check{\mathbf{S}}} : (\check{\mathbf{E}} - \tfrac{1}{2}(\nabla \mathbf{r}^{\mathrm{T}} \cdot \mathbf{g} \cdot \nabla \mathbf{r} - \nabla^0 \mathbf{r}^{\mathrm{T}} \cdot \mathbf{G} \cdot \nabla^0 \mathbf{r}) \\
& - \operatorname{sym}(\nabla \mathbf{r}^{\mathrm{T}} \cdot \mathbf{g} \cdot \nabla \mathbf{d} - \nabla^0 \mathbf{r}^{\mathrm{T}} \cdot \mathbf{G} \cdot \nabla^0 \mathbf{d})) \\
& - \dot{\tilde{\mathbf{S}}} \cdot (2\tilde{\mathbf{E}} - (\nabla \mathbf{r}^{\mathrm{T}} \cdot \mathbf{g} \cdot \mathbf{d}_{,3} - \nabla^0 \mathbf{r}^{\mathrm{T}} \cdot \mathbf{G} \cdot {}^0\mathbf{d}_{,3})) \\
& - \check{\mathbf{S}} : (\dot{\check{\mathbf{E}}} - \operatorname{sym}(\nabla \dot{\mathbf{r}}^{\mathrm{T}} \cdot \mathbf{g} \cdot \nabla \mathbf{r} + \nabla \dot{\mathbf{r}}^{\mathrm{T}} \cdot \mathbf{g} \cdot \nabla \mathbf{d} + \nabla \dot{\mathbf{d}}^{\mathrm{T}} \cdot \mathbf{g} \cdot \nabla \mathbf{r})) \\
& - \tilde{\mathbf{S}} \cdot (2\dot{\tilde{\mathbf{E}}} - (\nabla \dot{\mathbf{r}}^{\mathrm{T}} \cdot \mathbf{g} \cdot \mathbf{d}_{,3} + \dot{\mathbf{d}}_{,3} \cdot \mathbf{g} \cdot \nabla \mathbf{r})) \\
& - \theta \dot{\hat{\eta}} - \dot{\theta} \hat{\eta} + \hat{\mathbf{Q}} : \dot{\hat{\mathbf{P}}} + {}^p\theta \, {}^p\dot{\hat{\eta}} - \dot{\hat{\lambda}} \Phi \Bigg] \mathrm{d}\xi \, \mathrm{d}A
\end{aligned}$$

$$
-\int_{\omega} [\dot{\mathbf{p}} \cdot \mathbf{g} \cdot \mathbf{r} + \mathbf{p} \cdot \mathbf{g} \cdot \dot{\mathbf{r}}] \, \mathrm{d}A
$$

$$
-\int_{\Gamma_{\sigma}} [\dot{\hat{\mathbf{h}}} \cdot \mathbf{g} \cdot ((\mathbf{r} - \bar{\mathbf{r}}) + (\mathbf{d} - \bar{\mathbf{d}})) + \hat{\mathbf{h}} \cdot \mathbf{g} \cdot ((\dot{\mathbf{r}} - \dot{\bar{\mathbf{r}}}) + (\dot{\mathbf{d}} - \dot{\bar{\mathbf{d}}}))] \, \mathrm{d}\xi \, \mathrm{d}s, \tag{39}
$$

where $\bar{\boldsymbol{\phi}}$ as a variable has been replaced by $\{\mathbf{r}, \mathbf{d}\}$ and

$$
\mathbf{p} = \int_{-\varepsilon}^{+\varepsilon} \hat{\mathbf{f}} \, \mathrm{d}\xi + \hat{\mathbf{t}}_{-} + \hat{\mathbf{t}} \quad . \tag{40}
$$

Remarks:

(i) Note that all vector operators occurring in (39) and subsequently are now two dimensional, i.e., $\nabla \mathbf{v} = (v^{i}_{,\alpha})$, $i = 1, 2, 3$, $\alpha = 1, 2$.
(ii) There exist expressions for the internal energy where it is not possible to solve (35) for $\hat{E}_{33}$. It is then possible to keep $\hat{E}_{33}$ as an "internal variable" and still carry out the analysis to follow. Some expressions, however, will of course become more complicated. Also, the analogy to the three dimensional theory will be lost to a certain extent.

It is instructive to consider the state equations which have been altered by the substitutions above: variation of $\dot{\check{\mathbf{S}}}$ and $\dot{\tilde{\mathbf{S}}}$ gives the strains as

$$
\check{\mathbf{E}} = \tfrac{1}{2}(\nabla \mathbf{r}^{\mathrm{T}} \cdot \mathbf{g} \cdot \nabla \mathbf{r} - \nabla^{0} \mathbf{r}^{\mathrm{T}} \cdot \mathbf{G} \cdot \nabla^{0} \mathbf{r}) + \operatorname{sym}(\nabla \mathbf{r}^{\mathrm{T}} \cdot \mathbf{g} \cdot \nabla \mathbf{d} - \nabla^{0} \mathbf{r}^{\mathrm{T}} \cdot \mathbf{G} \cdot \nabla^{0} \mathbf{d}), \tag{41}
$$

and

$$
\tilde{\mathbf{E}} = \tfrac{1}{2}(\nabla \mathbf{r}^{\mathrm{T}} \cdot \mathbf{g} \cdot \mathbf{d}_{,3} - \nabla^{0} \mathbf{r}^{\mathrm{T}} \cdot \mathbf{G} \cdot {}^{0}\mathbf{d}_{,3}). \tag{42}
$$

Let us now introduce the unique decompositions

$$
\check{\mathbf{S}}(\zeta, \xi) = \frac{1}{2\varepsilon} \mathbf{S}^{0}(\zeta) + \bar{\mathbf{S}}(\zeta, \xi), \tag{43}
$$

with

$$
\int_{-\varepsilon}^{+\varepsilon} \bar{\mathbf{S}} \, \mathrm{d}\xi = 0 \tag{44}
$$

and

$$
\hat{\mathbf{h}}(\zeta, \xi) = \frac{1}{2\varepsilon} \mathbf{h}^{0}(\zeta) + \bar{\mathbf{h}}(\zeta, \xi), \tag{45}
$$

with

$$
\int_{-\varepsilon}^{+\varepsilon} \bar{\mathbf{h}} \, \mathrm{d}\xi = 0. \tag{46}
$$

Then, applying multiple partial integration, variation of $\dot{\mathbf{r}}$ yields

$$\nabla\cdot(\nabla\mathbf{r}\cdot\mathbf{S}^0)+\nabla\cdot\left[\int_{-\varepsilon}^{+\varepsilon}(\nabla\mathbf{d}\cdot\tilde{\mathbf{S}}+\mathbf{d}_{,3}\otimes\tilde{\mathbf{S}})\,\mathrm{d}\xi\right]+\mathbf{p}=0\quad\text{on}\,\omega \tag{47}$$

and

$$\mathbf{h}^0=\nabla\mathbf{r}\cdot\mathbf{S}^0\cdot\mathbf{n}+\left[\int_{-\varepsilon}^{+\varepsilon}(\nabla\mathbf{d}\cdot\tilde{\mathbf{S}}+\mathbf{d}_{,3}\otimes\tilde{\mathbf{S}})\,\mathrm{d}\xi\right]\cdot\mathbf{n}\quad\text{on}\,\gamma. \tag{48}$$

Variation of $\dot{\mathbf{d}}$ gives

$$\nabla\cdot(\nabla\mathbf{r}\cdot\tilde{\mathbf{S}})+\nabla\mathbf{r}\cdot\tilde{\mathbf{S}}_{,3}=0\quad\text{on}\,\Omega, \tag{49}$$

$$\tilde{\mathbf{h}}=\nabla\mathbf{r}\cdot\tilde{\mathbf{S}}\cdot\mathbf{n}\quad\text{on}\,\Gamma_o \tag{50}$$

and

$$\nabla\mathbf{r}\cdot\tilde{\mathbf{S}}=0\quad(\Leftrightarrow\tilde{\mathbf{S}}=0)\quad\text{on}\,\Gamma_+\cup\Gamma_-. \tag{51}$$

Note that the divergence operator "$\nabla\cdot$" in (47) and (49) is also two-dimensional, i.e., $\nabla\cdot\mathbf{T}=T_i{}^{\alpha}{}_{,\alpha}$ $\alpha=1,2$. The tensor or dyadic product of two vectors is defined by $(\mathbf{v}\otimes\mathbf{w})_{\alpha\beta}:=v_\alpha w_\beta$.

One observes that, particularly, (47) and (48) are not completely satisfactory. Property (51), however, is an important restriction which we will try to satisfy in an *a priori* manner in the further development of the theory.

Also, it should be emphasized that, in the asymptotic limit $\varepsilon\to 0$, the decomposition (43) becomes singular. As shown by Fox *et al.* (1993), the problem for $\mathbf{S}^0$ becomes decoupled in this case, constituting a membrane theory. Furthermore, they show that the form of the shell model obtained in the limit is strongly influenced by the assumed form of the external loading (in our case assumption (shell 1)). As already stated, we don't intend to proceed along this line of investigation within the present work.

5. BIORTHOGONAL FUNCTION SYSTEMS

We will call two sets of functions $\{\varphi^n\}$, $\{\psi^n\}$ defined on the interval $[-\varepsilon,+\varepsilon]$ biorthogonal if

$$(\varphi^m,\psi^n):=\int_{-\varepsilon}^{+\varepsilon}\varphi^m(\xi)\psi^n(\xi)\,\mathrm{d}\xi=\delta^{mn}, \tag{52}$$

where δ^{mn} denotes the Kronecker delta symbol.

Let f, g be two functions defined on $[-\varepsilon,+\varepsilon]$ and let us introduce the expansions

$$\hat{f}=f^n\varphi^n,\quad\hat{g}=g^n\psi^n, \tag{53}$$

where, because of (52), the expansion coefficients are given by

$$f^n=(f,\psi^n)=\int_{-\varepsilon}^{+\varepsilon}f\psi^n\,\mathrm{d}\xi,\quad g^n=(g,\varphi^n)=\int_{-\varepsilon}^{+\varepsilon}g\varphi^n\,\mathrm{d}\xi. \tag{54}$$

Furthermore, we have the important property

$$(\hat{f}, \hat{g}) = f^n g^n. \tag{55}$$

We call $\{\varphi^n\}$, $\{\psi^n\}$ complete if $f = \hat{f}$ and $g = \hat{g}$ for all square integrable functions f, $g \in \mathscr{L}^2[-\varepsilon, +\varepsilon]$. Of course it seems to be preferable to use expansions with respect to complete function systems. This is basically true, but there are situations where, e.g., an approximation by a finite set of functions like a collocation or a finite element ansatz may be more desirable (see Section 9.2).

6. REDUCTION TO A TWO DIMENSIONAL THEORY

Using expansions of the form (53) we will be able to carry out the "thickness integration" over ξ explicitly and thus obtain completely two dimensional shell theories.

As we will see immediately, the most favorable way to do this is to use expansions with respect to biorthogonal function systems for energetically conjugate quantities. Two quantities **A** and **B** are called energetically conjugate if $\mathbf{A} = \partial \hat{W}/\partial \mathbf{B}$.

Now let $\mathbf{A} = \mathbf{A}^n \varphi^n$, $\mathbf{B} = \mathbf{B}^n \psi^n$ and let us introduce the "integrated internal energy"

$$\bar{W} = \int_{-\varepsilon}^{+\varepsilon} \hat{W} \, \mathrm{d}\xi. \tag{56}$$

We get

$$\frac{\partial \bar{W}}{\partial \mathbf{B}^n} = \int_{-\varepsilon}^{+\varepsilon} \frac{\partial \hat{W}}{\partial \mathbf{B}} \psi^n \, \mathrm{d}\xi = \int_{-\varepsilon}^{+\varepsilon} \mathbf{A}\psi^n \, \mathrm{d}\xi = \mathbf{A}^n. \tag{57}$$

Hence we obtain a relation for the expansion coefficients $\mathbf{A}^n$, $\mathbf{B}^n$ which is analogous to that one for the quantities **A**, **B** themselves.

Motivated by this observation we introduce the following expansions. In order to have enough flexibility we will use different pairs of biorthogonal function systems, indicated by diverse subscripts, for the various pairs of energetically conjugate quantities.

Let us start with the deformation

$$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}^n \varphi^n, \quad \hat{\boldsymbol{\phi}}^n = (\hat{\boldsymbol{\phi}}, \psi^n), \tag{58}$$

where we make the special identification

$$\mathbf{r} = \hat{\boldsymbol{\phi}}^0, \quad \mathbf{d}^n = \hat{\boldsymbol{\phi}}^n, \quad n \geqslant 1. \tag{59}$$

This means we have

$$\varphi^0 = 1, \quad \psi^0 = \frac{1}{2\varepsilon}. \tag{60}$$

Moreover, we get

$$\mathbf{d} = \mathbf{d}^n \varphi^n, \quad n \geqslant 1\dagger. \tag{61}$$

for the director. Similar expansions will be assumed for ${}^0\boldsymbol{\phi}$ and $\hat{\bar{\boldsymbol{\phi}}}$.

The strains and stresses will be expanded according to

† Usually formulae will hold for indices ranging over the set $\{n \geqslant 0\}$. If this is not the case it will be explicitly stated.

$$\check{\mathbf{E}} = \mathbf{E}^n \varphi^n, \quad \tilde{\mathbf{E}} = \tilde{\mathbf{E}}^n \tilde{\varphi}^n, \tag{62}$$

$$\check{\mathbf{S}} = \mathbf{S}^n \psi^n, \quad \tilde{\mathbf{S}} = \tilde{\mathbf{S}}^n \tilde{\psi}^n. \tag{63}$$

For the remaining quantities we will assume

$$\hat{\eta} = \eta^n \varphi^n_\eta, \quad \theta = \theta^n \psi^n_\eta, \tag{64}$$

$$\hat{\mathbf{P}} = \mathbf{P}^n \varphi^n_p, \quad \hat{\mathbf{Q}} = \mathbf{Q}^n \psi^n_p, \tag{65}$$

$$^p\hat{\eta} = {}^p\eta^n \varphi^n_{p\eta}, \quad {}^p\theta = {}^p\theta^n \psi^n_{p\eta}, \tag{66}$$

$$\dot{\lambda} = \dot{\lambda}^n \varphi^n_\lambda, \quad \Phi = \Phi^n \psi^n_\lambda, \tag{67}$$

$$\hat{\mathbf{h}} = \mathbf{h}^n \psi^n. \tag{68}$$

In order to keep the theory concise, it is desirable to take only a few terms of the expansions (62) and (63) into account. This means that especially every summand in the expansion of the shear stress $\tilde{\mathbf{S}}$ should satisfy the boundary condition (51) separately. We achieve this by claiming

$$\tilde{\psi}^n(\pm\varepsilon) = 0 \quad \text{for all} \quad n \geqslant 0. \tag{69}$$

Substitution of (59) and (61)–(68) into (39) and carrying out the integration over ξ gives the completely two dimensional functional

$$\begin{aligned}
\dot{\Pi}_{UTEP} = \int_\omega \Bigg[& \frac{\partial \bar{W}}{\partial \mathbf{E}^n} : \dot{\mathbf{E}}^n + \frac{\partial \bar{W}}{\partial \tilde{\mathbf{E}}^n} \cdot \dot{\tilde{\mathbf{E}}}^n + \frac{\partial \bar{W}}{\partial \eta^n} \dot{\eta}^n + \frac{\partial \bar{W}}{\partial \mathbf{P}^n} : \dot{\mathbf{P}}^n + \frac{\partial \bar{W}}{\partial\, {}^p\eta^n} {}^p\dot{\eta}^n \\
& - \dot{\mathbf{S}}^0 : (\mathbf{E}^0 - \tfrac{1}{2}(\nabla \mathbf{r}^{\mathrm{T}} \cdot \mathbf{g} \cdot \nabla \mathbf{r} - \nabla^0 \mathbf{r}^{\mathrm{T}} \cdot \mathbf{G} \cdot \nabla^0 \mathbf{r})) \\
& - \dot{\mathbf{S}}^n : (\mathbf{E}^n - \mathrm{sym}(\nabla \mathbf{r}^{\mathrm{T}} \cdot \mathbf{g} \cdot \nabla \mathbf{d}^n - \nabla^0 \mathbf{r}^{\mathrm{T}} \cdot \mathbf{G} \cdot \nabla^0 \mathbf{d}^n)) \\
& - \dot{\tilde{\mathbf{S}}}^n \cdot (2\tilde{\mathbf{E}}^n - \kappa^{nk} (\nabla \mathbf{r}^{\mathrm{T}} \cdot \mathbf{g} \cdot \mathbf{d}^k - \nabla^0 \mathbf{r}^{\mathrm{T}} \cdot \mathbf{G} \cdot {}^0\mathbf{d}^k)) \\
& - \mathbf{S}^0 : (\dot{\mathbf{E}}^0 - \mathrm{sym}(\nabla \dot{\mathbf{r}}^{\mathrm{T}} \cdot \mathbf{g} \cdot \nabla \mathbf{r})) \\
& - \mathbf{S}^n : (\dot{\mathbf{E}}^n - \mathrm{sym}(\nabla \dot{\mathbf{r}}^{\mathrm{T}} \cdot \mathbf{g} \cdot \nabla \mathbf{d}^n + \nabla \dot{\mathbf{d}}^{nT} \cdot \mathbf{g} \cdot \nabla \mathbf{r})) \\
& - \tilde{\mathbf{S}}^n \cdot (2\dot{\tilde{\mathbf{E}}}^n - \kappa^{nk} (\nabla \dot{\mathbf{r}}^{\mathrm{T}} \cdot \mathbf{g} \cdot \mathbf{d}^k + \dot{\mathbf{d}}^k \cdot \mathbf{g} \cdot \nabla \mathbf{r})) \\
& - \theta^n \dot{\eta}^n - \dot{\theta}^n \eta^n + \mathbf{Q}^n : \dot{\mathbf{P}}^n + {}^p\theta^n\, {}^p\dot{\eta}^n - \dot{\lambda}^n \Phi^n - (\dot{\mathbf{p}} \cdot \mathbf{g} \cdot \mathbf{r} + \mathbf{p} \cdot \mathbf{g} \cdot \dot{\mathbf{r}}) \Bigg] \mathrm{d}A \\
& - \int_\gamma [\dot{\mathbf{h}}^0 \cdot \mathbf{g} \cdot (\mathbf{r} - \bar{\mathbf{r}}) + \dot{\mathbf{h}}^n \cdot \mathbf{g} \cdot (\mathbf{d}^n - \bar{\mathbf{d}}^n) \\
& + \mathbf{h}^0 \cdot \mathbf{g} \cdot (\dot{\mathbf{r}} - \dot{\bar{\mathbf{r}}}) + \mathbf{h}^n \cdot \mathbf{g} \cdot (\dot{\mathbf{d}} - \dot{\bar{\mathbf{d}}}^n)]\, \mathrm{d}s,
\end{aligned} \tag{70}$$

where we call

$$\kappa^{nk} = \left(\tilde{\psi}^n, \frac{\mathrm{d}}{\mathrm{d}\xi}\varphi^k\right) \tag{71}$$

the shear-correction-factor-matrix because as a special case it will contain the well known correction factor of Reissner-Mindlin-theory.

$\dot{\Pi}_{UTEP}$ is a functional with variables $\{\mathbf{E}^n, \tilde{\mathbf{E}}^n, \eta^n, \mathbf{S}^n, \tilde{\mathbf{S}}^n, \mathbf{r}, \mathbf{d}^n, \mathbf{h}^n, \mathbf{Q}^n, {}^p\theta^n, \dot{\mathbf{E}}^n, \dot{\tilde{\mathbf{E}}}^n, \dot{\eta}^n, \dot{\mathbf{S}}^n, \dot{\tilde{\mathbf{S}}}^n, \dot{\mathbf{r}}, \dot{\mathbf{d}}^n, \dot{\mathbf{h}}^n, \dot{\mathbf{P}}^n, {}^p\dot{\eta}^n\}$ and constant parameters $\{\mathbf{P}^n, {}^p\eta^n, \mathbf{p}, \bar{\mathbf{r}}, \bar{\mathbf{d}}^n, \theta^n, \dot{\mathbf{p}}, \dot{\bar{\mathbf{r}}}, \dot{\bar{\mathbf{d}}}^n, \dot{\lambda}^n, \dot{\theta}^n\}$.

Let us now investigate the stationarity conditions of $\dot{\Pi}_{UTEP}$. Variation with respect to $\dot{\mathbf{E}}^n$, $\dot{\tilde{\mathbf{E}}}^n$ and $\dot{\mathbf{P}}^n$, respectively, gives the constitutive laws

$$\mathbf{S}^n := \frac{\partial \bar{W}}{\partial \mathbf{E}^n}, \quad \tilde{\mathbf{S}}^n := \frac{1}{2}\frac{\partial \bar{W}}{\partial \tilde{\mathbf{E}}^n} \tag{72}$$

and

$$\mathbf{Q}^n := -\frac{\partial \bar{W}}{\partial \mathbf{P}^n}. \tag{73}$$

Variation with respect to $\dot{\eta}^n$ and ${}^p\dot{\eta}^n$, respectively yields

$$\theta^n = \frac{\partial \bar{W}}{\partial \eta^n}, \quad {}^p\theta^n = -\frac{\partial \bar{W}}{\partial\, {}^p\eta^n}. \tag{74}$$

Variation with respect to $\dot{\mathbf{S}}^n$ and $\dot{\tilde{\mathbf{S}}}^n$, respectively, gives the expressions for the strains

$$\mathbf{E}^0 = \tfrac{1}{2}(\nabla\mathbf{r}^{\mathrm{T}}\cdot\mathbf{g}\cdot\nabla\mathbf{r} - \nabla^0\mathbf{r}^{\mathrm{T}}\cdot\mathbf{G}\cdot\nabla^0\mathbf{r}), \tag{75}$$

$$\mathbf{E}^n = \operatorname{sym}(\nabla\mathbf{r}^{\mathrm{T}}\cdot\mathbf{g}\cdot\nabla\mathbf{d}^n - \nabla^0\mathbf{r}^{\mathrm{T}}\cdot\mathbf{G}\cdot\nabla^0\mathbf{d}^n), \quad n \geqslant 1, \tag{76}$$

$$\tilde{\mathbf{E}}^n = \tfrac{1}{2}\kappa^{nk}(\nabla\mathbf{r}^{\mathrm{T}}\cdot\mathbf{g}\cdot\mathbf{d}^k - \nabla^0\mathbf{r}^{\mathrm{T}}\cdot\mathbf{G}\cdot{}^0\mathbf{d}^k). \tag{77}$$

Variation with respect to $\dot{\mathbf{r}}$ gives the equilibrium condition

$$\nabla\cdot(\nabla\mathbf{r}\cdot\mathbf{S}^0) + \nabla\cdot(\nabla\mathbf{d}^n\cdot\mathbf{S}^n) + \kappa^{kn}(\nabla\cdot\tilde{\mathbf{S}}^k)\mathbf{d}^n + \mathbf{p} = 0 \tag{78}$$

and the boundary condition

$$\mathbf{h}^0 = \nabla\mathbf{r}\cdot\mathbf{S}^0\cdot\mathbf{n} + \nabla\mathbf{d}^n\cdot\mathbf{S}^n\cdot\mathbf{n} + \kappa^{kn}\mathbf{d}^n(\tilde{\mathbf{S}}^k\cdot\mathbf{n}) \quad \text{on}\,\gamma. \tag{79}$$

Variation with respect to $\dot{\mathbf{d}}^n$ gives the equilibrium conditions

$$\nabla\cdot(\nabla\mathbf{r}\cdot\mathbf{S}^n) + \kappa^{kn}\nabla\mathbf{r}\cdot\tilde{\mathbf{S}}^k = 0, \quad n \geqslant 1 \tag{80}$$

and the boundary conditions

$$\mathbf{h}^n = \nabla\mathbf{r}\cdot\mathbf{S}^n\cdot\mathbf{n}, \quad n \geqslant 1, \quad \text{on}\,\gamma. \tag{81}$$

Variation with respect to $\dot{\mathbf{h}}^n$ yields the boundary conditions

$$\mathbf{r} = \bar{\mathbf{r}}, \quad \mathbf{d}^n = \bar{\mathbf{d}}^n, \quad n \geqslant 1, \quad \text{on}\,\gamma. \tag{82}$$

Finally, variation with respect to $\mathbf{Q}^n$ and ${}^p\theta^n$, respectively, gives the flow rules (evolution laws)

$$\dot{\mathbf{P}}^n = \dot{\lambda}^k \frac{\partial \Phi^k}{\partial \mathbf{Q}^n}, \quad {}^p\dot{\eta}^n = \dot{\lambda}^k \frac{\partial \Phi^k}{\partial {}^p\theta^n}. \tag{83}$$

Once again, variation with respect to $\{\mathbf{E}^n, \tilde{\mathbf{E}}^n, \eta^n, \mathbf{S}^n, \tilde{\mathbf{S}}^n, \mathbf{r}, \mathbf{d}^n, \mathbf{h}^n, {}^p\theta^n\}$ yields the time derivatives of the equations derived above.

We would like to point out now the strict analogy between the two-dimensional equations (72)–(83) just derived and the three-dimensional ones given by (15)–(22), (41), (42) and (47)–(50). ((51) is satisfied implicitly now.)

7. THE INEQUALITY CONSTRAINTS

We still have to discuss the subsidiary conditions (23) in order to complete the two-dimensional theory. In the general case this turns out to be a quite complicated task involving convex analysis, see, e.g., Rockafellar (1974). To simplify matters we will introduce the following assumption:

A set of functions $\{\varphi^n\}$ possesses the separation property if, for any $f(\xi) = f^n\varphi^n(\xi)$, the following holds:

$$f(\xi) \geqslant 0 \;\forall \xi \Leftrightarrow f^n \geqslant 0 \;\forall n. \tag{84}$$

Consider for example the following collocation ansatz:

Choose $\xi^0, \ldots, \xi^N, \hat{\xi}^0, \ldots, \hat{\xi}^{N+1} \in [-\varepsilon, +\varepsilon]$ such that

$$-\varepsilon \leqslant \hat{\xi}^0 \leqslant \xi^0 < \hat{\xi}^1 < \xi^1 < \cdots < \xi^{N-1} < \hat{\xi}^N < \xi^N \leqslant \hat{\xi}^{N+1} \leqslant +\varepsilon \tag{85}$$

and define

$$\varphi^n_\lambda(\xi) := \delta(\xi - \xi^n), \quad \psi^n_\lambda(\xi) := \chi[\hat{\xi}^n, \hat{\xi}^{n+1}], \tag{86}$$

where δ is the Dirac-delta-function and $\chi[\xi, \eta]$ denotes the characteristic function of the interval $[\xi, \eta]$. Clearly both $\{\varphi^n_\lambda\}$ and $\{\psi^n_\lambda\}$ possess the separation property.

Another example is given by

$$\varphi^n_\lambda(\xi) := \frac{1}{\hat{\xi}^{n+1} - \hat{\xi}^n} \chi[\hat{\xi}^n, \hat{\xi}^{n+1}], \quad \psi^n_\lambda(\xi) := \chi[\hat{\xi}^n, \hat{\xi}^{n+1}]. \tag{87}$$

If $\{\varphi^n_\lambda\}$ and $\{\psi^n_\lambda\}$ possess the separation property, the inequalities (23) decouple into

$$\Phi^n \leqslant 0, \quad \dot{\lambda}^n \geqslant 0, \quad \dot{\lambda}^{(n)}\Phi^{(n)} = 0.\dagger \tag{88}$$

8. THE ELASTOPLASTIC TANGENT

For numerical purposes the elastoplastic tangent operator, i.e., the relation between stress and strain rates is of special importance. Let us introduce the abbreviations $\underline{\mathbf{E}}^n := \{\mathbf{E}^n, \tilde{\mathbf{E}}^n, \eta^n\}$, $\underline{\mathbf{P}}^n := \{\mathbf{P}^n, {}^p\eta^n\}$, $\underline{\mathbf{S}}^n := \{\mathbf{S}^n, \tilde{\mathbf{S}}^n, \theta^n\}$, $\underline{\mathbf{Q}}^n := \{\mathbf{Q}^n, {}^p\theta^n\}$.

Variation of $\underline{\mathbf{E}}^n$ and $\underline{\mathbf{P}}^n$ in (70) gives the constitutive relations for the time rates

$$\dot{\underline{\mathbf{S}}}^n = \frac{\partial^2 \bar{W}}{\partial \underline{\mathbf{E}}^n \partial \underline{\mathbf{E}}^k} : \dot{\underline{\mathbf{E}}}^k + \frac{\partial^2 \bar{W}}{\partial \underline{\mathbf{E}}^n \partial \underline{\mathbf{P}}^k} : \dot{\underline{\mathbf{P}}}^k, \tag{89}$$

† There is no summation over indices in parentheses here and subsequently.

$$\dot{\underline{\mathbf{Q}}}^n = -\frac{\partial^2 \bar{W}}{\partial \underline{\mathbf{P}}^n \, \partial \underline{\mathbf{E}}^k} : \dot{\underline{\mathbf{E}}}^k - \frac{\partial^2 \bar{W}}{\partial \underline{\mathbf{P}}^n \, \partial \underline{\mathbf{P}}^k} : \dot{\underline{\mathbf{P}}}^k, \tag{90}$$

where of course partial derivatives and the ":"-product have to be defined respecting the actual structure of the quantities involved. The flow rules (83) now become

$$\dot{\underline{\mathbf{P}}}^n = \dot{\lambda}^k \frac{\partial \Phi^k}{\partial \underline{\mathbf{Q}}^n}. \tag{91}$$

In order to proceed further we have to define the set of active yield surfaces

$$\text{Act} := \{n, \dot{\lambda}^n > 0\}. \tag{92}$$

For plastic loading ($\dot{\lambda}^n > 0$, i.e., $n \in \text{Act}$) we have from (88) $\Phi^n = 0$ during the whole loading process. Hence, we get the plastic consistency condition

$$\dot{\Phi}^{\underline{n}} = -\frac{\partial \Phi^{\underline{n}}}{\partial \underline{\mathbf{Q}}^k} : \left(\frac{\partial^2 \bar{W}}{\partial \underline{\mathbf{P}}^k \, \partial \underline{\mathbf{E}}^l} : \dot{\underline{\mathbf{E}}}^l + \frac{\partial^2 \bar{W}}{\partial \underline{\mathbf{P}}^k \, \partial \underline{\mathbf{P}}^l} : \dot{\underline{\mathbf{P}}}^l \right) + \frac{\partial \Phi^{\underline{n}}}{\partial \underline{\mathbf{P}}^k} : \dot{\underline{\mathbf{P}}}^k = 0.\dagger \tag{93}$$

Substitution of (91) into (93) and solving for $\dot{\lambda}^{\underline{n}}$ gives

$$\dot{\lambda}^{\underline{n}} = (\underline{K}^{-1})^{nk} \frac{\partial \Phi^{\underline{k}}}{\partial \underline{\mathbf{Q}}^l} : \frac{\partial^2 \bar{W}}{\partial \underline{\mathbf{P}}^l \, \partial \underline{\mathbf{E}}^m} : \dot{\underline{\mathbf{E}}}^m, \tag{94}$$

with

$$\underline{\mathbf{K}}^{\underline{nk}} := -\frac{\partial \Phi^{\underline{n}}}{\partial \underline{\mathbf{Q}}^l} : \frac{\partial^2 \bar{W}}{\partial \underline{\mathbf{P}}^l \, \partial \underline{\mathbf{P}}^m} : \frac{\partial \Phi^{\underline{k}}}{\partial \underline{\mathbf{Q}}^m} + \frac{\partial \Phi^{\underline{n}}}{\partial \underline{\mathbf{P}}^l} : \frac{\partial \Phi^{\underline{k}}}{\partial \underline{\mathbf{Q}}^l}. \tag{95}$$

Substitution of (91) and (94) into (89) finally yields

$$\dot{\underline{\mathbf{S}}}^n = {}^{ep}\underline{\mathbf{D}}^{nk} : \dot{\underline{\mathbf{E}}}^k, \tag{96}$$

where the elastoplastic tangent operator ${}^{ep}\underline{\mathbf{D}}^{nk}$ is given by

$${}^{ep}\underline{\mathbf{D}}^{nk} = \frac{\partial^2 \bar{W}}{\partial \underline{\mathbf{E}}^n \, \partial \underline{\mathbf{E}}^k} - (\underline{K}^{-1})^{\underline{ij}} \frac{\partial^2 \bar{W}}{\partial \underline{\mathbf{E}}^n \, \partial \underline{\mathbf{P}}^l} : \frac{\partial \Phi^{\underline{i}}}{\partial \underline{\mathbf{Q}}^l} \otimes \frac{\partial \Phi^{\underline{j}}}{\partial \underline{\mathbf{Q}}^m} : \frac{\partial^2 \bar{W}}{\partial \underline{\mathbf{P}}^l \, \partial \underline{\mathbf{E}}^k}. \tag{97}$$

Here once again the tensor product "$\otimes$" has to be understood in the appropriate way.

From (95) one clearly sees that in general $\underline{K}$ is non-symmetric, i.e., $\underline{K}^{nk} \neq \underline{K}^{kn}$. Hence this is also true for ${}^{ep}\underline{\mathbf{D}}$. It is, however, symmetric if Φ does not explicitly depend on $\mathbf{P}$, i.e., if $\partial\Phi/\partial\mathbf{P} = 0$.

9. A SHELL MODEL FOR LARGE STRAIN ISOTHERMAL ELASTOPLASTICITY

9.1. *Kinematics and equilibrium conditions*

In the following we will give a specific example for the theory developed so far. The model is valid for a moderately thick shell subjected to large elastoplastic strains. In order

† Here and further on, underlined indices range over Act only.

to keep matters simple we will disregard thermal effects but it is of course possible to include those in a straightforward way.

In this subsection we will consider the expansions of the kinematic quantities and the stresses. If we assume the shell to be shallow in its initial configuration then expansions up to order $n = 1$ are sufficient even for high curvatures in the actual configuration. Let us therefore set

$$\varphi^0 = 1, \quad \varphi^1 = \xi, \quad \tilde{\varphi}^0 = \varepsilon^2 - \xi^2, \tag{98}$$

$$\psi^0 = \frac{1}{2\varepsilon}, \quad \psi^1 = \frac{3}{2\varepsilon^3}\xi, \quad \tilde{\psi}^0 = \frac{15}{16\varepsilon^5}(\varepsilon^2 - \xi^2). \tag{99}$$

Clearly, $\tilde{\psi}^0$ satisfies (69). This ansatz corresponds to Reissner-Mindlin theory. The only nonvanishing shear correction factor is then

$$\kappa^{01} = \frac{5}{4\varepsilon^2}. \tag{100}$$

The director is given by

$$\mathbf{d} = \mathbf{d}^1\xi \tag{101}$$

and the strains are

$$\hat{E}_{\alpha\beta} = (E^0)_{\alpha\beta} + (E^1)_{\alpha\beta}\xi, \quad \hat{E}_{\alpha 3} = \hat{E}_{3\alpha} = (\tilde{E}^0)_\alpha(\varepsilon^2 - \xi^2). \tag{102}$$

The stresses become

$$\hat{S}^{\alpha\beta} = \frac{1}{2\varepsilon}(S^0)^{\alpha\beta} + \frac{3}{2\varepsilon^3}(S^1)^{\alpha\beta}\xi, \quad \hat{S}^{\alpha 3} = \hat{S}^{3\alpha} = \frac{15}{16\varepsilon^5}(\tilde{S}^0)^\alpha(\varepsilon^2 - \xi^2). \tag{103}$$

Of course, $\mathbf{S}^0$ now denotes the normal forces, $\mathbf{S}^1$ the bending moments and $\tilde{\mathbf{S}}^0$ the shearing forces inside the shell.

We obtain the following definitions for the strains:

$$(E^0)_{\alpha\beta} = \tfrac{1}{2}(\mathbf{r}_{,\alpha}\cdot\mathbf{g}\cdot\mathbf{r}_{,\beta} - {}^0\mathbf{r}_{,\alpha}\cdot\mathbf{g}\cdot{}^0\mathbf{r}_{,\beta}), \tag{104}$$

$$(E^1)_{\alpha\beta} = \tfrac{1}{2}(\mathbf{r}_{,\alpha}\cdot\mathbf{g}\cdot\mathbf{d}^1_{,\beta} + \mathbf{r}_{,\beta}\cdot\mathbf{g}\cdot\mathbf{d}^1_{,\alpha} - {}^0\mathbf{r}_{,\alpha}\cdot\mathbf{g}\cdot{}^0\mathbf{d}^1_{,\beta} - {}^0\mathbf{r}_{,\beta}\cdot\mathbf{g}\cdot{}^0\mathbf{d}^1_{,\alpha}), \tag{105}$$

$$(\tilde{E}^0)_\alpha = \frac{5}{4\varepsilon^2}\mathbf{r}_{,\alpha}\cdot\mathbf{g}\cdot\mathbf{d}^1. \tag{106}$$

Finally we get the following equilibrium conditions:

$$(\mathbf{r}_{,\alpha}(S^0)^{\alpha\beta})_{,\beta} + (\mathbf{d}^1_{,\alpha}(S^1)^{\alpha\beta})_{,\beta} + \frac{5}{4\varepsilon^2}\mathbf{d}^1(\tilde{S}^0)^\alpha_{,\alpha} + \mathbf{p} = 0, \tag{107}$$

$$(\mathbf{r}_{,\alpha}(S^1)^{\alpha\beta})_{,\beta} + \frac{5}{4\varepsilon^2}\mathbf{r}_{,\alpha}(\tilde{S}^0)^\alpha = 0. \tag{108}$$

The boundary conditions (81) and (82) remain unaltered and will not be repeated here.

9.2. *Constitutive theory*

We will use a constitutive model which is valid for moderately large elastic strains up to about 0.2. This is sufficient to describe even deformations of the shell resulting in high curvatures (up to $1/4\varepsilon$).

Before we can start deriving two-dimensional stress-strain-relations for shell theory we have to agree on expressions for the three-dimensional internal energy W and yield function Φ. For isotropic large strain elastoplasticity it is suggested in Hackl (1995), that the internal energy depends solely on the invariants of the elastic right Cauchy-Green-tensor

$$ {}^{e}\bar{\mathbf{C}} = {}^{e}\mathbf{F}^{\mathrm{T}} \cdot \mathbf{g} \cdot {}^{e}\mathbf{F}. \tag{109}$$

We will base our analysis on compressible neo-Hookean material, see, e.g., Ciarlet (1988), and set

$$ W = U({}^{e}j) + \tfrac{1}{2}\mu \operatorname{tr}(\bar{\mathbf{G}}^{-1} \cdot {}^{e}\bar{\mathbf{C}}), \tag{110}$$

where

$$ {}^{e}j = \sqrt{\det(\bar{\mathbf{G}}^{-1} \cdot {}^{e}\bar{\mathbf{C}})} \tag{111}$$

and

$$ U({}^{e}j) = \tfrac{1}{4}\lambda\, {}^{e}j^{2} - \tfrac{1}{2}(\lambda + 2\mu) \log {}^{e}j. \tag{112}$$

Here, λ and μ are Lamé-parameters.†

Remark: W, as defined above, satisfies three fundamental requirements for large-strain internal energy functions:

(i) It reduces to the expression for linear isotropic elasticity in the small strain limit, i.e., for $\bar{\mathbf{G}}^{-1} \cdot {}^{e}\bar{\mathbf{C}}$ close to identity.
(ii) It holds $U({}^{e}j) \to \infty$ for ${}^{e}j \to \infty$ or ${}^{e}j \to 0$. In particular, compression to zero volume requires infinite force.
(iii) $\partial W/\partial^{e}\mathbf{C} = 0$ for $\bar{\mathbf{G}}^{-1} \cdot {}^{e}\bar{\mathbf{C}} = \bar{\mathbf{I}}$ (identity on the intermediate configuration), i.e., the stress is zero if there is no deformation.

Because tensor invariants remain unchanged by pull-back it holds that

$$ \operatorname{tr}(\bar{\mathbf{G}}^{-1} \cdot {}^{e}\bar{\mathbf{C}}) = \operatorname{tr}({}^{p}\mathbf{C}^{-1} \cdot \mathbf{C}) \tag{113}$$

and

$$ {}^{e}j = {}^{p}j^{-1} j, \tag{114}$$

where

$$ \mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{g} \cdot \mathbf{F} = \mathbf{G} + 2\mathbf{E} \tag{115}$$

denotes the total right Cauchy-Green-tensor and

$$ {}^{p}\mathbf{C} = {}^{p}\mathbf{F}^{\mathrm{T}} \cdot \bar{\mathbf{G}} \cdot {}^{p}\mathbf{F} \tag{116}$$

† Note that the Lamé-parameter λ is not related in any way to the plastic consistency parameter $\dot{\lambda}$. Knowing that there is a danger of confusion we still have chosen to keep up this notation in order to be consistent with the common usage in the literature.

the plastic right Cauchy-Green-tensor, and we have introduced the determinants

$$j = \sqrt{\det(\mathbf{G}^{-1} \cdot \mathbf{C})} \tag{117}$$

and

$${}^{p}j = \sqrt{\det(\mathbf{G}^{-1} \cdot {}^{p}\mathbf{C})}. \tag{118}$$

We will assume the plastic flow to be isochoric, i.e., it holds that ${}^{p}j = 1$ for all time. Thus the internal energy assumes the form

$$W = \tfrac{1}{4}\lambda j^2 - \tfrac{1}{2}(\lambda + 2\mu)\log j + \tfrac{1}{2}\mu\,{}^{p}\mathbf{C}^{-1} : \mathbf{C}. \tag{119}$$

Using

$$\frac{\partial \det(\mathbf{A})}{\partial \mathbf{A}} = \det(\mathbf{A})\mathbf{A}^{\mathrm{T}-1} \tag{120}$$

we get

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} = \left(\frac{\lambda}{2}j^2 - \frac{\lambda + 2\mu}{2}\right)\mathbf{C}^{-1} + \mu\,{}^{p}\mathbf{C}^{-1} \tag{121}$$

for the stress tensor.

At this point we will introduce a slight modification to the scheme developed in the previous sections. It turns out to be favorable to be able to expand j independently of $\mathbf{C}$. Therefore we would like to treat j as a separate variable. This can be achieved by adding the definition (117) to the functional (14) via a Lagrange-multiplier. That is, we introduce a modified internal energy

$$V := W - \pi\left(j - \sqrt{\det(\mathbf{G}^{-1} \cdot \mathbf{C})}\right) \tag{122}$$

and new variables j, π and $\dot{j}$, $\dot{\pi}$, respectively. π can be viewed as a stress like variable (a hydrostatic pressure). With those definitions (121) decouples into

$$\mathbf{S} = \frac{\partial V}{\partial \mathbf{E}} = \pi\sqrt{\det(\mathbf{G}^{-1} \cdot \mathbf{C})}\,\mathbf{C}^{-1} + \mu\,{}^{p}\mathbf{C}^{-1} \tag{123}$$

and

$$\pi = \frac{\partial U}{\partial j} = \frac{1}{2}\lambda j - \frac{1}{2}(\lambda + 2\mu)\frac{1}{j}. \tag{124}$$

Next we choose $\mathbf{P} := \frac{1}{2}{}^{p}\mathbf{C}^{-1}$ as the internal parameter. Then the thermodynamically associated force becomes

$$\mathbf{Q} = -\frac{\partial W}{\partial \mathbf{P}} = -\mu\mathbf{C}. \tag{125}$$

A von Mises-type yield function is given by

$$\Phi = \bar{\Phi}(\mathbf{Q}, \mathbf{P}) := \|\operatorname{dev}({}^{p}\mathbf{C}^{-1} \cdot \mathbf{Q})\|^2 - R^2 = \|\operatorname{dev}(\mathbf{S} \cdot \mathbf{C})\|^2 - R^2. \tag{126}$$

$R = \sqrt{2/3}\sigma_Y$, where σ_Y denotes the yield stress. The norm and the deviator of a mixed-variant tensor $\mathbf{T}$ are defined in the usual way by $\|\mathbf{T}\| := \sqrt{\mathbf{T} : \mathbf{T}^{\mathrm{T}}}$ and $\mathrm{dev}(\mathbf{T}) := \mathbf{T} - \frac{1}{3}\mathrm{tr}(\mathbf{T})\mathbf{I}$, respectively.

The flow rules become now

$$\dot{\mathbf{P}} = \dot{\lambda}\frac{\partial \Phi}{\partial \mathbf{Q}} = 2\dot{\lambda}\,\mathrm{dev}({}^p\mathbf{C}^{-1} \cdot \mathbf{Q}) \cdot {}^p\mathbf{C}^{-1}. \tag{127}$$

Using (120) we finally get

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(\mathbf{G}^{-1} \cdot {}^p\mathbf{C}) = \det(\mathbf{G}^{-1} \cdot {}^p\mathbf{C})\,\mathrm{tr}({}^p\mathbf{C}^{-1} \cdot {}^p\dot{\mathbf{C}}) = 0 \tag{128}$$

because of (127). Hence, our formulation is consistent with the assumption ${}^pj = 1$.

An overview of the whole constitutive model is provided by Table 1.

According to Section 3, the next step is to pull back the formulation onto Ω. Let us therefore introduce quantities

$${}^0\mathbf{C} := {}^0\mathbf{F}^{\mathrm{T}} \cdot \mathbf{G} \cdot {}^0\mathbf{F}, \quad \hat{\mathbf{C}} := {}^0\mathbf{F}^{\mathrm{T}} \cdot \mathbf{C} \cdot {}^0\mathbf{F} = {}^0\mathbf{C} + 2\hat{\mathbf{E}} \tag{129}$$

and

$${}^p\hat{\mathbf{C}} := {}^0\mathbf{F}^{\mathrm{T}} \cdot {}^p\mathbf{C} \cdot {}^0\mathbf{F}, \quad \hat{\mathbf{P}} := {}^0\mathbf{F}^{-1} \cdot \mathbf{P} \cdot {}^0\mathbf{F}^{\mathrm{T}-1}. \tag{130}$$

The yield function now simply becomes

$$\Phi = 4\,{}^0j^{-2}\|\mathrm{dev}(\hat{\mathbf{P}} \cdot \hat{\mathbf{Q}})\|^2 - R^2. \tag{131}$$

As for the internal energy matters are more complicated: let us assume additionally that $\|\hat{\mathbf{E}}\| \ll 1$, i.e., that the shear-deformations remain moderate. Then it holds that

$$\det(\mathbf{G}^{-1} \cdot \mathbf{C}) = \frac{1}{{}^0j^2}\det(\hat{\mathbf{C}}) = \frac{\hat{C}_{33}}{{}^0j^2}\det(\check{\mathbf{C}}), \tag{132}$$

where we make use of the decomposition (35), (36). Note that once again the expressions $\det(\hat{\mathbf{C}})$ and $\det(\check{\mathbf{C}})$ are only defined via the Cartesian structure of Ω.

Table 1. Three-dimensional constitutive model

Internal energy:	$V = \frac{1}{4}\lambda j^2 - \frac{1}{2}(\lambda + 2\mu)\log j + \mu\mathbf{P} : \mathbf{C}$ $- \pi(j - \sqrt{\det(\mathbf{G}^{-1} \cdot \mathbf{C})})$
Yield function:	$\Phi = 4\|\mathrm{dev}(\mathbf{P} \cdot \mathbf{Q})\|^2 - R^2$
Extra variables:	$\pi = \frac{1}{2}\lambda j - \frac{1}{2}(\lambda + 2\mu)\frac{1}{j}, \quad j = \sqrt{\det(\mathbf{G}^{-1} \cdot \mathbf{C})}$
Stresses:	$\mathbf{S} = \pi\sqrt{\det(\mathbf{G}^{-1} \cdot \mathbf{C})}\,\mathbf{C}^{-1} + 2\mu\mathbf{P}$
Thermodynamically associated forces:	$\mathbf{Q} = -\mu\mathbf{C}$
Flow rule:	$\dot{\mathbf{P}} = 8\dot{\lambda}\,\mathrm{dev}(\mathbf{P} \cdot \mathbf{Q}) \cdot \mathbf{P}$

Decomposing $\hat{\mathbf{P}}$ according to (36) and using (132) we get

$$\hat{V} = {}^0jV = \frac{\hat{\lambda}}{4}{}^0jj^2 - \frac{\hat{\lambda}+2\mu}{2}{}^0j\log j + {}^0j\mu(\check{\mathbf{P}}:\check{\mathbf{C}} + 2\tilde{\mathbf{P}}\cdot\tilde{\mathbf{C}} + \hat{P}^{33}C_{33}) - {}^0jj\pi + \hat{C}_{33}^{1/2}\sqrt{\det(\check{\mathbf{C}})}\,\boldsymbol{\pi}. \tag{133}$$

Solving of $\partial\hat{V}/\partial\hat{\mathbf{E}}_{33} = 0$ for $\hat{C}_{33}^{1/2}$ yields

$$\hat{C}_{33}^{1/2} = -\frac{\sqrt{\det(\check{\mathbf{C}})}\pi}{2\,{}^0j\mu\hat{P}^{33}}. \tag{134}$$

Substitution into (133) gives

$$\hat{V} = \frac{\hat{\lambda}}{4}{}^0jj^2 - \frac{\hat{\lambda}+2\mu}{2}{}^0j\log j + {}^0j\mu(\check{\mathbf{P}}:\check{\mathbf{C}} + 2\tilde{\mathbf{P}}\cdot\tilde{\mathbf{C}}) - {}^0jj\pi - \frac{\det(\check{\mathbf{C}})\pi^2}{4\,{}^0j\mu\hat{P}^{33}}. \tag{135}$$

The resulting constitutive model is displayed in Table 2.

Note that π does not play the role of a Lagrange-multiplier anymore. The state equations for j and π have now become coupled.

The next step is the derivation of the two dimensional model according to Section 6. For this purpose we have to make a choice on the function systems $\{\varphi_p^n\}$ and $\{\psi_p^n\}$. We will assume both systems to consist of piecewise linear functions. Let $\xi^0, \ldots, \xi^N$ be collocation points as in (85). Let ψ_p^n be the piecewise linear function with $\psi_p^n(\xi^n) = 1$ and $\psi_p^n(\xi^i) = 0$ for $i \neq n$. Because φ_p^n is piecewise linear as well, there is a unique expansion

$$\varphi_p^n = b^{nk}\psi_p^k. \tag{136}$$

Table 2. Constitutive model on the set Ω

Internal energy:

$$\hat{V} = \frac{\hat{\lambda}}{4}{}^0jj^2 - \frac{\hat{\lambda}-2\mu}{2}{}^0j\log j + {}^0j\mu(\check{\mathbf{P}}:\check{\mathbf{C}} + 2\tilde{\mathbf{P}}\cdot\tilde{\mathbf{C}}) - {}^0jj\pi - \frac{\det(\check{\mathbf{C}})\pi^2}{4\,{}^0j\mu\hat{P}^{33}}$$

Yield function:

$$\Phi = 4\,{}^0j^{-2}\,|\mathrm{dev}(\hat{\mathbf{P}}\cdot\hat{\mathbf{Q}})|^2 - R^2$$

Extra variables:

$$\pi = \frac{1}{2}\hat{\lambda}j - \frac{1}{2}(\hat{\lambda}+2\mu)\frac{1}{j}, \quad j = -\frac{\det(\check{\mathbf{C}})\pi}{2\,{}^0j^2\mu\hat{P}^{33}}$$

Stresses:

$$\check{\mathbf{S}} = -\frac{\det(\check{\mathbf{C}})\pi^2}{2\,{}^0j\mu\hat{P}^{33}}\check{\mathbf{C}}^{-1} + 2\,{}^0j\mu\check{\mathbf{P}}$$

$$\left(= -\frac{\hat{\lambda}+2\mu}{2}\frac{{}^0j}{1-\frac{\hat{\lambda}}{\mu}\det(\check{\mathbf{C}})\,({}^0j^2\hat{P}^{33})}\check{\mathbf{C}}^{-1} - 2\,{}^0j\mu\check{\mathbf{P}}\right)$$

$$\tilde{\mathbf{S}} = 2\,{}^0\mathrm{j}\mu\tilde{\mathbf{P}}$$

Thermodynamically associated forces:

$$\check{\mathbf{Q}} = -{}^0j\mu\check{\mathbf{C}}, \quad \tilde{\mathbf{Q}} = -{}^0j\mu\tilde{\mathbf{C}}, \quad \hat{Q}_{33} = \frac{\det(\check{\mathbf{C}})\pi^2}{4\,{}^0j\mu(\hat{P}^{33})^2}$$

Flow rule:

$$\dot{\hat{\mathbf{P}}} = 8\,{}^0j^{-2}\hat{\lambda}\,\mathrm{dev}(\hat{\mathbf{P}}\cdot\hat{\mathbf{Q}})\cdot\hat{\mathbf{P}}$$

It is easy to see that $\{\varphi_p^n\}$ and $\{\psi_p^n\}$ are biorthogonal if and only if

$$b = c^{-1}, \tag{137}$$

where c is the matrix defined by

$$c^{nk} := (\psi_p^n, \psi_p^k). \tag{138}$$

$\{\varphi_p^n\}$ and $\{\psi_p^n\}$ are depicted in Fig. 1 for the case of 6 collocation points ($N = 5$). Only half of the basis functions are shown; the remaining ones (for $n = 3, 4, 5$) are mirror images of those.

Note that $\{\varphi_p^n\}$ as well as $\{\psi_p^n\}$ form a basis for all piecewise linear functions with knots $\{\xi^n\}$, i.e., we have for any such function f

$$(f, \varphi_p^n) = f(\xi^n). \tag{139}$$

We get the expansions (65) which we repeat here for reference:

$$\hat{\mathbf{P}} = \mathbf{P}^n \varphi_p^n, \quad \hat{\mathbf{Q}} = \mathbf{Q}^n \psi_p^n. \tag{140}$$

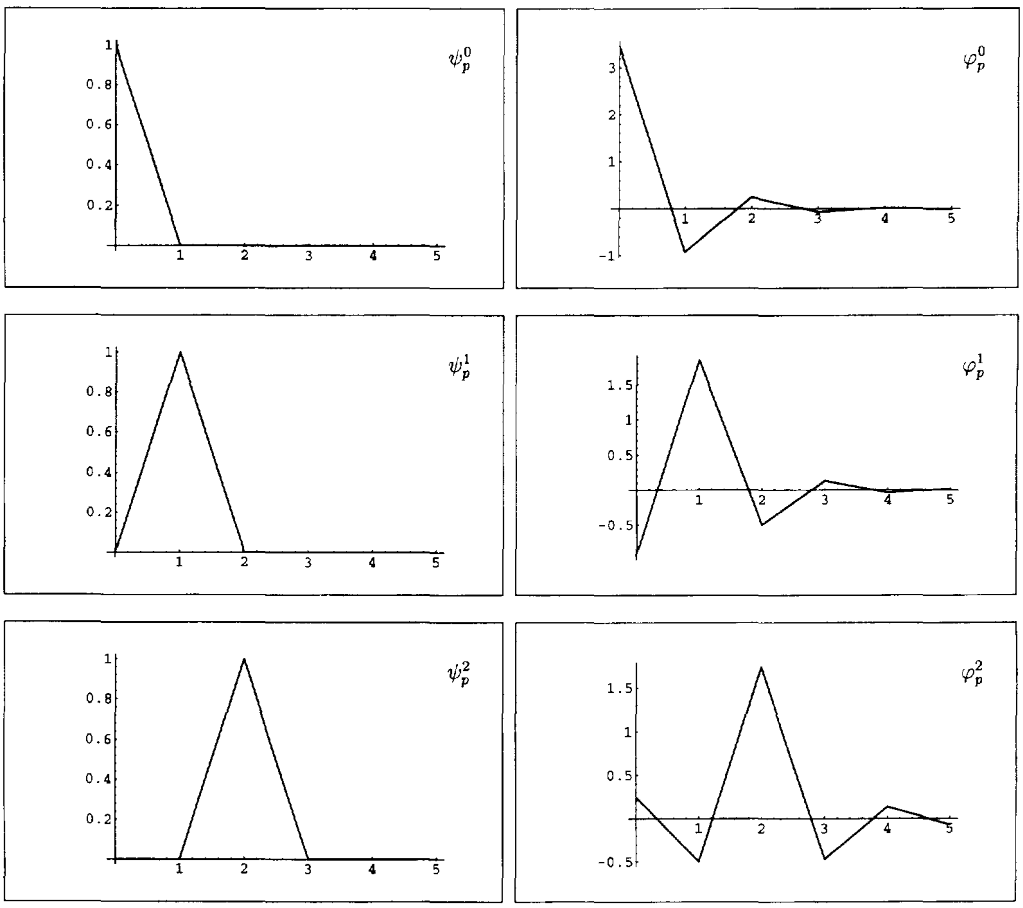

Fig. 1. Piecewise linear biorthogonal function systems.

(140) splits into

$$\check{\mathbf{P}} = \check{\mathbf{P}}^n \varphi_p^n, \quad \tilde{\mathbf{P}} = \tilde{\mathbf{P}}^n \varphi_p^n, \tag{141}$$

$$\check{\mathbf{Q}} = \check{\mathbf{Q}}^n \psi_p^n, \quad \tilde{\mathbf{Q}} = \tilde{\mathbf{Q}}^n \psi_p^n. \tag{142}$$

We will assume the quantities $\hat{P}^{33}$ and $\hat{Q}_{33}$ are constant across the shell thickness. Because of (136), (137) and (139) this yields the following relations for the expansion coefficients:

$$(P^n)^{33} = \hat{P}^{33} \sum_k c^{nk}, \quad (Q^n)_{33} = \hat{Q}_{33}. \tag{143}$$

Moreover, we define

$$a^n := (\varphi_p^n, \tilde{\varphi}^0). \tag{144}$$

For the variables j and π, we assume the expansions

$$j = j^0 \varphi^0 + j^1 \varphi^1, \quad \pi = \pi^0 \psi^0 + \pi^1 \psi^1. \tag{145}$$

At this point we have to utilize once more the restriction that the initial configuration of the shell possesses only moderate curvature. This means that ${}^0j$ is approximately constant over the shell thickness and given by

$${}^0j = \frac{h(\zeta)}{2\varepsilon} \|{}^0\mathbf{r}_{,1} \wedge {}^0\mathbf{r}_{,2}\|. \tag{146}$$

Moreover, we have

$$\det(\check{\mathbf{C}}) = \det(\mathbf{C}^1)\xi^2 + \mathbf{C}^1 \overset{\wedge}{\wedge} \mathbf{C}^0 \xi + \det(\mathbf{C}^0), \tag{147}$$

where the wedge- and double-wedge-product of two matrices of rank two are given by $(\mathbf{A} \wedge \mathbf{B})_{\alpha\beta} := \varepsilon_{\gamma\delta} A_{\alpha\gamma} B_{\delta\beta}$ and $(\mathbf{A} \overset{\wedge}{\wedge} \mathbf{B})_{\alpha\beta} := \varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} A_{\alpha\gamma} B_{\beta\delta}$.

Finally we are going to choose the collocation ansatz (86) for $\{\varphi_\lambda^n\}$ and $\{\psi_\lambda^n\}$.

With all those expansions introduced now we are able to perform the integration over the shell thickness in a straightforward way. We will not go into any details but display the results in Table 3.

Note that many peculiarities of the three-dimensional model are still reflected by the two-dimensional one. There is, however, one distinct difference: It is no longer possible in the two-dimensional model to eliminate the extra variables j^0, j^1, π^0, π^1 *ab initio*. They have to be retained and solved for together with the other unknowns.

10. CONCLUSIONS

A flexible framework for nonlinear shells has been established which allows to categorize almost every existing specific theory (analytical as well as numerical in nature) in a unified manner. The formulation is able to encompass general anisotropic thermo-elastoplastic material behavior and has been worked out in detail for the isothermal and isotropic case. Moreover, it can be extended in a straightforward way to include rate-dependent and damage models. The approach yields analytical models which exhibit a close relationship to the underlying three-dimensional theory.

Future research should discuss more closely the role of the *a priori* assumptions used. Also, applications of the theory presented to materials involving volumetric-deviatoric splits or anisotropy are desirable. Finally the shell models thus obtained as well as the example presented in Section 9 in this paper will have to be implemented numerically.

Table 3. Two-dimensional constitutive model

Internal energy:

$$\bar{V} = \frac{\lambda}{4}{}^0j\left[2\varepsilon(j^0)^2 + \frac{2}{3}\varepsilon^3(j^1)^2\right]$$

$$- \frac{\lambda+2\mu}{2}{}^0j\left[\varepsilon\log((j^0)^2-(\varepsilon j^1)^2) + 2\frac{j^0}{j^1}\tanh^{-1}\left(\frac{\varepsilon j^1}{j^0}\right)\right]$$

$$+ {}^0j\mu\left[\sum_n \check{\mathbf{P}}^n : \mathbf{C}^0 + \zeta^n\check{\mathbf{P}}^n : \mathbf{C}^1 + 2a^n\tilde{\mathbf{P}}^n \cdot \tilde{\mathbf{C}}^0\right] - {}^0jj^n\pi^n$$

$$- \frac{1}{4{}^0j\mu\hat{P}^{33}}\left[\det(\mathbf{C}^0)((\pi^0)^2+(\pi^1)^2) + \frac{2}{\sqrt{3}}\varepsilon\mathbf{C}^0 \,\hat{\wedge}\, \mathbf{C}^1\pi^0\pi^1\right.$$

$$\left. + \varepsilon^2\det(\mathbf{C}^1)\left(\frac{1}{3}(\pi^0)^2 + \frac{3}{5}(\pi^1)^2\right)\right]$$

Yield function:

$$\Phi^{(i)} = 4{}^0j^{-2}(b^{(i)k}b^{(i)l}\operatorname{dev}(\mathbf{P}^k \cdot \mathbf{Q}^{(i)}) : \operatorname{dev}(\mathbf{P}^l \cdot \mathbf{Q}^{(i)})^{\mathrm{T}}) - R^2$$

Extra variables:

$$\pi^0 = \lambda\varepsilon j^0 - (\lambda+2\mu)\frac{1}{j^1}\tanh^{-1}\left(\frac{\varepsilon j^1}{j^0}\right)$$

$$\pi^1 = \frac{\lambda}{3}\varepsilon^3 j^1 - (\lambda+2\mu)\left(\frac{\varepsilon}{j^1} - \frac{j^0}{(j^1)^2}\tanh^{-1}\left(\frac{\varepsilon j^1}{j^0}\right)\right)$$

$$j^0 = \frac{1}{4{}^0j^2\mu\hat{P}^{33}}\left[\left(2\det(\mathbf{C}^0) + \frac{2}{3}\varepsilon^2\det(\mathbf{C}^1)\right)\pi^0 + \frac{2}{\sqrt{3}}\varepsilon\mathbf{C}^0 \,\hat{\wedge}\, \mathbf{C}^1\pi^1\right]$$

$$j^1 = \frac{1}{4{}^0j^2\mu\hat{P}^{33}}\left[\frac{2}{\sqrt{3}}\varepsilon\mathbf{C}^0 \,\hat{\wedge}\, \mathbf{C}^1\pi^0 + \left(2\det(\mathbf{C}^0) + \frac{5}{6}\varepsilon^2\det(\mathbf{C}^1)\right)\pi^1\right]$$

Stresses:

$$\mathbf{S}^0 = -\frac{1}{2{}^0j\mu\hat{P}^{33}}\left[\det(\mathbf{C}^0)((\pi^0)^2+(\pi^1)^2)(\mathbf{C}^0)^{-1}\right.$$

$$\left. + \frac{2}{\sqrt{3}}\varepsilon\det(\mathbf{C}^1)\pi^0\pi^1(\mathbf{C}^1)^{-1}\right] + 2{}^0j\mu\sum_n\check{\mathbf{P}}^n$$

$$\mathbf{S}^1 = -\frac{1}{2{}^0j\mu\hat{P}^{33}}\left[\frac{2}{\sqrt{3}}\varepsilon\det(\mathbf{C}^0)\pi^0\pi^1(\mathbf{C}^0)^{-1}\right.$$

$$\left. + \varepsilon^2\det(\mathbf{C}^1)\left(\frac{1}{3}(\pi^0)^2 + \frac{3}{5}(\pi^1)^2\right)(\mathbf{C}^1)^{-1}\right] + 2{}^0j\mu\zeta^n\check{\mathbf{P}}^n$$

$$\tilde{\mathbf{S}}^0 = 2{}^0j\mu a^n\tilde{\mathbf{P}}^n$$

Thermodynamically associated forces:

$$\mathbf{Q}^n = -{}^0j\mu(\mathbf{C}^0 + \zeta^n\mathbf{C}^1)$$

$$\tilde{\mathbf{Q}}^n = -2{}^0j\mu a^n\tilde{\mathbf{C}}^0$$

$$\hat{Q}_{33} = -\frac{1}{4{}^0j\mu(\hat{P}^{33})^2}\left[\det(\mathbf{C}^0)((\pi^0)^2+(\pi^1)^2) + \frac{2}{\sqrt{3}}\varepsilon\mathbf{C}^0 \,\hat{\wedge}\, \mathbf{C}^1\pi^0\pi^1\right.$$

$$\left. + \varepsilon^2\det(\mathbf{C}^1)\left(\frac{1}{3}(\pi^0)^2 + \frac{3}{5}(\pi^1)^2\right)\right]$$

Flow rule:

$$\dot{\mathbf{P}}^{(n)} = 8\dot{\lambda}^{(n)0}j^{-2}b^{(n)k}b^{(n)l}\operatorname{dev}(\mathbf{P}^k \cdot \mathbf{Q}^{(n)}) \cdot \mathbf{P}^l$$

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